ON THE CAUCHY PROBLEM FOR HYPERBOLIC DIFFERENTIAL EQUATIONS WITH MULTIPLE CHARACTERISTICS

BY

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ABSTRACT

A priori estimates leading to existence results for the Cauchy problem are obtained for a class of linear variable coefficient hyperbolic operators with multiple characteristics. These estimates provide an extension of the energy inequalities known for strictly hyperbolic operators.

The Cauchy problem for general linear variable coefficient hyperbolic differential equations has been studied much more thoroughly in the case of operators which are strictly hyperbolic i.e., have simple real characteristics than in the case of operators with multiple characteristics. A. Lax [5] has studied hyperbolic equations with multiple characteristics involving one space variable. Ohya [6] has obtained the existence and uniqueness of solutions in certain Gevrey spaces for hyperbolic operators with coefficients in Gevrey spaces, such that the multiplicities of the characterist'cs are invariant.

In this paper we shall extend the energy inequality method to obtain a priori estimates and thus existence results for certain classes of hyperbolic operators with multiple characteristics. Some techniques similar to those developed for operators of constant strength will be used. Our methods differ from those of [5], [6] so that the class of operators considered here is neither included in nor includes those considered in [5], [6].

Using standard multi-index notation, let $P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$ denote a linear operator of order m whose coefficients $a_\alpha(x)$ are defined on an open subset Ω of R_n containing 0. The sum $P_m(x,D) = \sum_{|\alpha|=m} a_\alpha(x)D^\alpha$ then denotes the principle part of $P(x,D)$, and $P_m(x,\xi)$ the corresponding homogeneous poly-

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nomial. $P_m(x,D)$ is called hyperbolic with respect to the vector $N = (0,\cdots,0,1) \in R_n$ at the point $x \in \Omega$ if the roots $r_j(x,\xi')$, $j = 1,\cdots,m$ of $P_m(x,\xi',r) = 0$ are real, $\xi' \in R_{n-1}$[1]. We shall assume the following type of constant strength condition regarding these roots.

(A) There exists a constant $C > 0$ such that

$$\sum_{k=1}^{m} \prod_{j\neq k} |\zeta_n - r_j(x,\xi')|^2 \leqq C \sum_{k=1}^{m} \prod_{j\neq k} |\zeta_n - r_j(0,\xi')|^2 ,$$

$$\xi' \in R_{n-1}, \ \zeta_n \in C, \ x \in \Omega.$$

We shall also assume the following factorization.

$$\text{(B)} \qquad P_m(x,\xi) = \prod_{i=1}^{l} P_{(i)}(x,\xi), \ \xi \in R_n, \ x \in \Omega,$$

where each $P_{(i)}(x,\xi)$ is a homogeneous hyperbolic polynomial with coefficients in $C^1(\Omega)$ which is either strictly hyperb·lic or of order 2.

THEOREM 1. *If conditions* (A), (B) *are valid for the hyperbolic operator* $P_m(x,D)$ *with coefficients in* $C^1(\Omega)$, *then*

(C) *There exists positive constants* δ, τ_0, C_1, C_2 *such that*

$$\tau^{2m} \int_I |u|^2 \exp(2\tau x_n)\, dx \leqq C_1 \tau^2 \int_I R^n(x,D,\bar{D}) u\bar{u} \exp(2\tau x_n)\, dx$$

$$\leqq C_2 \int_I |P_m(x,D)u|^2 \exp(2\tau x_n)\, dx, \ u \in C_0^\infty(\Omega_\delta), \ \tau \geqq \tau_0 .$$

Here $\Omega_\delta = \{x \in R_n : |x| < \delta\}$, $I = \{x \in R_n : x_n > 0\}$, and $R^n(x,D,\bar{D})u\bar{u}$ denotes a certain sum, $\sum_{\alpha,\beta}(x) D^\alpha u D^\beta u$, to be defined in §1.

We can also consider lower order terms. Let $P(x,D) = P_m(x,D) + \sum_{k=\sigma}^{m-1} L_k(x,D)$ where L_k is homogeneous of order k with coefficients in $L^\infty(\Omega)$.

COROLLARY 1. *Let* $P(x,D)$ *denote an operator whose principle part satisfies the conditions of Theorem* 1. *If there exists a* $C > 0$ *such that for* $x \in \Omega$. $\xi' \in R_{n-1}$, *and* $\zeta_n \in \mathbb{C}$,

$$\text{(D)} \qquad \sum_{k=0}^{m-1} \tau^{2(m-k-1)} |L_k(x,\xi',\zeta_n)|^2 \leqq C \sum_{k=1}^{m} \prod_{j\neq k} |\zeta_n - r_j(x,\xi')|^2,$$

then the estimates (C) *are valid for* $P(x,D)$.

We remark that (D) implies that $P(x,D)$ is hyperbolic by Lemma. 1.1 below and Theorem 5.5 of [2]. It is not known whether, conversely, every hyperbolic operator satisfies (D). For some related questions, see [4, pp. 135–136].

In proving Theorem 1, we shall obtain an estimate (1.9) which implies (A) for $x \in \Omega_\delta \cap I$. The proof of this necessity of (A) is similar to the proof of Theorem 8.1.1 of [4] and will be omitted.

It seems likely that the estimates (C) can be used to obtain the existence of C^∞ solutions of the Cauchy problem for the adjoint differential operator ${}^tP(x,D)$ by suitably extending some of the material in Ch. II and IX of [4] and applying a functional analysis result, Theorem 5.1, of [2]. We shall not develop such C^∞ results here, but briefly point out an L^2 existence result. Let $L_I^2(\Omega_\delta)$ denote the closed subspace of $L^2(\Omega_\delta)$ consisting of those functions f such that $\operatorname{supp} f \subset \bar{I}$.

THEOREM 2. *If ${}^tP(x,D)$ is a hyperbolic operator satisfying the conditions of Theorem* 1 *or Corollary* 1, *then for every $f \in L_I^2(\Omega_\delta)$, there is a solution $u \in L_1^2(\Omega_\delta)$ such that $P(x,D)u = f$ in Ω.*

Proof. The dual space $L_I^2(\Omega_\delta)'$ of $L_I^2(\Omega_\delta)$ is the space of L^2 functions defined on $\Omega_\delta \cap I$. The subspace $C_{(0)}^\infty(\Omega_\delta) = \{u \in C_0^\infty(\Omega_\delta),\ u \text{ restricted to } \bar{I}\}$ is dense in $L_I^2(\Omega_\delta)'$. Consider the closure T of the mapping ${}^tP(x,D)$: $C_{(0)}^\infty(\Omega_\delta) \cap L_I^2(\Omega_\delta)') \to L_I^2(\Omega_\delta)'$. T is densely defined, and the estimates (C) imply that T has a continuous inverse. Therefore $T' : L^2{}_I(\Omega_\delta) \to L_I^2(\Omega_\delta)$, which is defined by $P(x,D)$, is surjective.

1. This section is concerned with the proof of Theorem 1. We shall assume throughout that $P(x,D) = P_m(x,D)$, and that $P(x,D)$ is hyperbolic with respect to $N = (0,\cdots,0,1)$. Then $P(x,D)$ is proportional to an operator with real coefficients [1]. Since a constant of proportionality does not affect the validity of (C), we can assume that $P(x,D)$ itself has real coefficients. Let $d\mu(\tau)$ denote $e^{2\tau x_n}\,dx$. The energy integral,

$$E(u) = \int_I 2\operatorname{Im} P(x,D)u\,\overline{P^n(x,D)u}\,d\mu(\tau) \tag{1.1}$$

will be integrated by parts. This will yield a non-negative boundary term plus two sums of terms $R(u)$ and $H(u)$ such that $H(u)$ can be estimated by $R(u)$.

For the integration by parts, we employ the differential quadratic forms developed in [3, 4]. Such a form $F(x,D,\bar{D})u\bar{u} = \sum_{\alpha,\beta} a_{\alpha,\beta}(x) D^\alpha u \overline{D^\alpha u}$, with coefficients $a_{\alpha\beta} \in C^1(\Omega)$, can be integrated by parts i.e. there exists forms $G^k(x,D,\bar{D})u\bar{u}$

such that $F(x,D,\bar{D})u\bar{u} = -\sum_{k=1}^{n}\partial/\partial x_k\, G^k(x,D,\bar{D})u\bar{u} + \sum_{k=1}^{n} G_{(k)}^k(x,D,\bar{D})u\bar{u}$ iff the corresponding polynomial $F(x,\zeta,\bar{\zeta})$, $\zeta \in \mathbf{C}_n$, satisfies $F(x,\xi,\xi) = 0$, $\xi \in R_n$. In this case,

$$(1.2)\qquad F(x,\zeta,\bar{\zeta}) = -i\sum_{k=1}^{n}(\zeta_k - \overline{\zeta_k})G^k(x,\zeta,\bar{\zeta}),\quad \zeta = (\zeta_1,\cdots,\zeta_n)\in \mathbf{C}_n.$$

Here $G_{(k)}^k(x,\zeta,\bar{\zeta}) = \dfrac{\partial}{\partial x_k}G^k(x,\zeta,\bar{\zeta})$.

If $F(x,D,\bar{D})u\bar{u} = 2\,\mathrm{Im}\,P(x,D)u\overline{P^n(x,D)u}$, then clearly $F(x,\xi,\xi) = 0$, $\xi \in R_n$. In fact, the forms $G^k(x,\zeta,\bar{\zeta})$ can be chosen to be homogeneous of order $m-1$, see [3, p. 243]. Denoting G^k by R^k in this case, we have $E(u)$

$$= -\int_I \sum_{k=1}^{n}\frac{\partial}{\partial x_k}R\,(x,D,\bar{D})u\bar{u}d\mu(\tau)$$

$$+\int_I \sum_{k=1}^{n} R_{(k)}^k(x,D,\bar{D})u\bar{u}d\mu(\tau)\qquad \text{for } u\in C_0^\infty(\Omega).$$

Since there is only one boundary term, an integration by parts now gives.

$$(1.3)\quad E(u) = 2\tau\int_I R^n(x,D,\bar{D})u\bar{u}d\mu(\tau)$$

$$+\int_{x_n=0} R^n(x,D,\bar{D})u\bar{u}dx' + \sum_{k=1}^{n}\int_I R_{(k)}^k(x,D,\bar{D})u\bar{u}\,d\mu(\tau)\cdot$$

For a differential quadratic form $F(x,D,\bar{D})u\bar{u}$, let $F(x';D,\bar{D})u\bar{u}$, $x'\in\Omega$, denote the form with constant coefficients obtained by fixing the coefficients of $F(x,\bar{D},D)u\bar{u}$ at $x = x'$. Our first step in estimating the terms of (1.3) is the following estimate

THEOREM 1.1. *Condition* (A) *implies that there is a positive, continuous function* $C(x')$ *such that*

$$(1.4)\qquad \int_I R^n(x';D,\bar{D})u\bar{u}d\mu(\tau) \leqq C(x')\int_I R^n(0;D,\bar{D})u\bar{u}d\mu(\tau)$$

$$u\in C_0^\infty(R_n),\quad \tau>0.$$

The proof will be given by several lemmas.

LEMMA 1.1. *If* $P(x,D)$ *is hyperbolic with respect to* $N = (0,\cdots,0,1)$ *and homogeneous of order* m, *then for* $\xi'\in R_{n-1}$, $\zeta_n = \xi_n + i\tau \in \mathbf{C}_1$, $x\in\Omega$,

$$(1.5)\qquad R^n(x;\xi',\zeta_n,\xi',\bar{\zeta}_n) = \tau^{-1}\,\mathrm{Im}\,P(x,\xi',\zeta_n)\overline{P^n(x,\xi',\zeta_n)}$$

$$= P(x,N)^2 \sum_{k=1}^{m} \prod_{j \neq k} |\zeta_n - r_j(x,\xi')|^2 .$$

Proof. The equality of the first two terms is clear by substituting $\zeta = (\xi', \zeta_n)$ into (1.2). Now $\partial/\partial\tau |P(x,\xi',\xi_n + i\tau)|^2 = 2 \operatorname{Im} P(x,\xi',\zeta_n) P^n(x,\xi',\zeta_n)$. But $P(x,\xi',\xi_n + i\tau) = P(x,N) \prod_{j=1}^{m} (\xi_n + i\tau - r_j(x,\xi'))$, so this derivative also equals 2τ times the third term in (1.5).

In obtaining estimates from algebraic inequalities, we shall use the fact that for any form with constant coefficients,

$$(1.6) \qquad \int_I G(D,\bar{D}) u\bar{u} \, d\mu(\tau) = (2\pi)^{n-1} \int_0^\infty \int_{R_{n-1}} G(\xi', D_n + i\tau, \xi', \overline{D_n + i\tau}) v_n \bar{v}_n d\xi' dx_n$$

for all $u \in C_0^\infty(R_n)$, where $v(x) = u(x)e^{\tau x_n}$ and $v_n(\xi', x_n)$ is the Fourier transform of v with respect to the variables $x_1, \cdots, x_{n-1}$. The verification of (1.6) is essentially the same as that of formula (9.1.6) of [4] since

$$G(D,\bar{D}) u\bar{u} e^{2\tau x_n} = G(D', D_n + i\tau, \overline{D', D_n + i\tau}) v\bar{v} .$$

LEMMA 1.2. *Let $a_1, \cdots, a_{m-1}$, $b_1, \cdots, b_m \in \mathbb{C}$. If*

$$(1.7) \qquad \left| \prod_{j=1}^{m-1} (\zeta - a_j) \right|^2 \leq C \sum_{k=1}^{m} \left| \prod_{j \neq k} (\zeta - b_j) \right|^2, \quad \zeta \in \mathbb{C},$$

then there exists coefficients c_k, $k = 1, \cdots, m$, such that $|c_k| \leq \sqrt{C}$ and

$$(1.8) \qquad \prod_{j=1}^{m-1} (\zeta - a_j) = \sum_{k=1}^{m} c_k \prod_{j \neq k} (\zeta - b_j), \qquad \zeta \in \mathbb{C}.$$

Proof. Suppose there are l different b_j, say b_i', $i = 1, \cdots, l$, with b_i' occurring m_i times. Then (1.7) implies that for each i, at least $m_i - 1$ of the a_j must equal b_i'. Reorder the a_j so that $a_1, \cdots, a_{l-1}$ denote the remaining a_j. Cancelling $| \prod_{i=1}^{l} (\xi - b_i')^{m_i - 1} |^2$ from (1.7), one obtains

$$\left| \prod_{j=1}^{l-1} (\zeta - a_j) \right|^2 \leq C \sum_{i=1}^{l} m_i \left| \prod_{j \neq i} (\zeta - b_j') \right|^2 .$$

By the Lagrange interpolation formula,

$$\prod_{j=1}^{l-1} (\zeta - a_j) = \sum_{i=1}^{l} p_i \prod_{j=i}^{l} (\zeta - b_j') \quad \text{where } p_i = \prod_{j=i}^{l-1} (b_i' - a_j) \prod_{j \neq i} (b_i' - b_j')^{-1}.$$

Multiplying each side by $\prod_{i=1}^{l} (\zeta - b_i')^{mj-1}$ now yields (1.8) where $c_k = p_i / m_i$ for some i. Thus by the above estimate, $|c_k| = m_i^{-1} |p_i| \leq \sqrt{C}$.

We can now complete the proof of Theorem 1.1. By formulas (1.5) and (1.6), the left side of (1.4) is an integral of $\sum_{k=1}^{m}\left|\prod_{j\neq k}(D_n + i\tau - r_j(x', \xi'))\hat{v}_n\right|^2$. For a fixed $k = h$, consider $Q^h(x';D) = \prod_{j\neq h}(D_n + i\tau - r_j(x',\xi'))$. Condition (A) implies that

$$P(x,N)^2 \sum_{k=1}^{m} \left|Q^k(x';\zeta)\right|^2 \leqq P(0,N)^2 \sum_{k=1}^{m} \left| Q^k(0;\zeta)\right|^2.$$

Thus if we choose $a_j = -i\tau + r_j(x',\xi')$, $b_j = -i\tau + r_j(0,\xi')$, Lemma 1.2 implies that $Q^h(x',\zeta) = \sum_{k=1}^{m} c_k Q^j(0;\zeta)$ where $|c_k| \leqq P(0,N)P(x',N)^{-1}\sqrt{C}$. Thus $\left|\prod_{j\neq h}(D_n + i\tau - r_j(x',\xi'))\hat{v}_n\right|^2 = \left| \sum_{k=1}^{m} c_k Q^k(0;D_n)\hat{v}_n\right|^2 \leqq CP(x', \xi')^{-2}R^n(0; \xi', D_n + i\tau, \xi', \overline{D_n + i\tau})\hat{v}_n\bar{v}_n$. Summing over $h = 1,\cdots, n$ and integrating we obtain (1.4).

We next proceed in a manner suggested by arguments used for operators of constant strength. The class of forms $G(\zeta,\bar{\zeta})$ which are homogeneous in ζ and $\bar{\zeta}$ of order $m-1$ is clearly a finite dimensional vector space W_{m-1}. Let $R_0(u) = \int_I R^n(0;D,\bar{D})u\bar{u}\,d\mu(\tau)$, and let W denote that subspace of $G(D,\bar{D})u\bar{u} \in W_{m-1}$ such that for some $C > 0$, $\int_I |G(D,\bar{D})u\bar{u}|\,d\mu(\tau) \leqq CR_0(u)$. W then has a finite basis of forms $\{H_j\}_{j=1}^{l}$.

Lemma 1.3. *If $F(x,D,\bar{D})u\bar{u}$ is a form such that $F(x';D,\bar{D})u\bar{u} \in W$ for every $x' \in \Omega$ then for any $\Omega' \Subset \Omega$, there is a $C > 0$ such that for $u \in C_0^\infty(\Omega')$*

$$\int_I |F(x,D,\bar{D})u\bar{u}|\,d\mu(\tau) \leqq CR_0(u).$$

$$\int_I |F_{(k)}(x, D,\bar{D})u\bar{u}|\,d\mu(\tau) \leqq CR_0(u), \quad k = 1,\cdots n.$$

Proof. We may write $F(x,D,\bar{D})u\bar{u} = \sum_{j=1}^{l} c_j(x)H_j(D,\bar{D})u\bar{u}$ where $c_j(x) \in C^1(\Omega)$ since we have made a change of basis. Thus

$$F_{(k)}(x,D,\bar{D})u\bar{u} = \sum_{j=1}^{l} \frac{\partial c_j}{\partial x_j}(x)H_j(D\bar{D})u\bar{u}.$$

The estimates follow immediately from these representations.

Let $R(u) = \int_I R^n(x,D,\bar{D})u\bar{u}d\mu(\tau)$. Lemma 1.3 implies that for any $\Omega' \Subset \Omega$, there is a $C > 0$ such that

$$R(u) \leqq CR_0(u), \quad u \in C_0^\infty(\Omega'). \tag{1.9}$$

We shall also need an opposite estimate.

LEMMA 1.4. *There exist positive constants* δ, C *such that* $R_0(u) \leqq CR(u)$ *for all* $u \in C_0^\infty(\Omega_\delta)$ *where* $\Omega_\delta = \{x \in R_n : |x| < \delta\}$.

Proof. $R^n(0;D,\bar{D})u\bar{u} - R^n(x';D,\bar{D})u\bar{u} \in W$, $x' \in \Omega$, so this difference equals $\sum_{j=1}^{l} c_j(x')H_j$ where $c_j(0) = 0$, $j = 1,\cdots,l$. Thus

$$R_0(u) \leqq R(u) + M_\delta \sum_{j=1}^{l} \int_I |H_j(D,\bar{D})u\bar{u}|\, d\mu(\tau), \quad u \in C_0^\infty(\Omega_\delta),$$

where $M_\delta = \max\{|c_j(x)| : |x| < \delta,\ j = 1,\cdots,l\}$. Choosing δ small enough, we can transpose the last term.

Consider next the boundary term $\int_{x_n=0} R^n(x,D,\bar{D})u\bar{u}dx'$ of (1.3). For fixed $x' \in \Omega$, $\int_{x_n=0} R^n(x';D,\bar{D})u\bar{u}dx = (2\pi)^{n-1}\int R^n(x';\xi',D_n,\xi',\bar{D}_n)\hat{v}_n\bar{\hat{v}}_n d\xi'$. Thus we can treat this boundary term in a manner analogous to the above treatment of the main term, $R(u)$, obtaining the analogue of Lemma 1.4 in particular. For our purposes, it suffices to deduce from this that the boundary term is non-negative, at least for u having sufficiently small support. Thus (1.3) leads to the following estimate for $u \in C_0^\infty(\Omega_\delta)$.

$$2\tau R(u) + \sum_{k=1}^{n} \int_I R^k_{(k)}(x,D,\bar{D})u\bar{u}d\mu(\tau) \leqq E(u). \tag{1.10}$$

It remains essentially to estimate the middle term of (1.10) which will be denoted by $H(u)$. This can be done when the following condition is satisfied.

There exists $C > 0$ such that for $x' \in \Omega$, $k = 1,\cdots,n$,

$$\int_I |R^k(x';D,\bar{D})u\bar{u}|\, d\mu(\tau) \leqq C\int_I R^n(x';D,\bar{D})u\bar{u}d\mu(\tau), \quad u \in C_0^\infty(R_n). \tag{1.11}$$

THEOREM 1.2. *Let* $P(x,D)$ *be a hyperbolic operator, homogeneous of degree* m. *Then* (A) *and* (1.11) *imply the estimates* (C).

Proof. Since $R^n_{x'} \in W$, condition (1.11) implies that $R_{x'} \in W$, $k = 1,\cdots,n$, $x' \in \Omega$. Thus by Lemma 1.3, the middle terms $H(u)$ of (1.10) can be estimated by $R_0(u)$. By Lemma 1.4, this implies that for some $C' > 0$, $|H(u)| \leqq C'R(u)$, $u \in C_0^\infty(\Omega_\delta)$. Hence, $2\tau R(u) + H(u) \geqq 2\tau R(u) - |H(u)| \geqq R(u)$ for $\tau \geqq C'$. Choosing $\tau_0 = C'$, we obtain now from (1.10) that $\tau R(u) \leqq E(u)$, $u \in C_0^\infty(\Omega_\delta)$, $\tau \geqq \tau_0$.

Formula (1.5) implies that $\tau^{2(m-1)} \leqq P(0,N)^{-2}R^n(0;\xi',\zeta_n,\xi',\bar{\zeta}_n)$, $\zeta_n = \xi_n + i\tau$, so by (1.6), $\tau^{2(m-1)} \int_I |u|^2 d\mu(\tau) \leqq CR_0(u)$. Thus for $\tau \geqq \tau_0$, $u \in C_0(\Omega_\delta)$, (1.12) $$\tau^{2m}\int_I |u|^2 d(\tau) \leqq C_1\tau^2 R_0(u) \leqq C_1\tau^2 R_0(u) \leqq C_2\tau^2 R(u) \leqq C_2\tau E(u).$$

Now $\partial/\partial\tau P(x',\xi_n + i\tau)$ equals $iP(x',N)\ \Sigma_{k=1}^n \prod_{j\neq k}(\xi_n + i\tau - r_j(x',\xi'))$ and $iP^n(x',\xi_n + i\tau)$, so $|P^n(x',\xi' + i\tau)| \leqq CR^n(x';\xi',\zeta_n,\xi_n,\bar\zeta_n) \leqq R^n(0;\xi',\zeta_n,\xi',\bar\zeta_n)$. Applying (1.6) to the form $|P^n(x';D)u|^2$ with constant coefficients, we deduce that it is in the space W. Thus by Lemma 1.3,

$$\int_I |P^n(x,D)u|^2 d\mu(\tau) \leqq CR_0(u),\ u \in C_0^\infty(\Omega_\delta).$$

Finally, by the Cauchy-Schwartz inequality,

(1.13) $$\tau E(u) \leqq \varepsilon^{-1}\int_I |P(x,D)u|^2 d\mu(\tau) + \varepsilon\tau^2\int_I |P^n(x,D)u|^2 d\mu(\tau),\ \varepsilon > 0.$$

Estimating the last term of (1.13) by $\varepsilon\tau^2 R_0 u)$, and choosing ε sufficiently small we obtain (C) from (1.12) and (1.13).

2. It remains to show that (B) implies (1.11) and to prove Corollary 1. We begin by considering second order operators $P(x,D)$. We can then write $P(x,D) = \Sigma_{i,j=1}^n a_{ijc}x)D_iD_j$ where the matrix (a_{ij}) is real symmetric. Let $P(x,\zeta,\bar\zeta)$ denote $\Sigma_{i,j=1}^n a_{ij}(x)\zeta_i\bar\zeta_j,\ \zeta \in \mathbf{C}_n$.

LEMMA 2.1. *If $P(x,D)$ is of order 2, then there is a $C > 0$ such that*

$$|P^k(x;\xi',\zeta_n)|^2 \leqq CR^n(x;\xi',\zeta_n,\xi',\bar\zeta_n), \qquad k = 1,\cdots,n.$$

Proof. For P_m hyperbolic, (5.5.3) of [4] directly implies that for each α,

(2.1) $$\tau^{2(m-|\alpha|)}|P_m^\alpha(x;\xi',\zeta_n)|^2 \leqq C|P_m(x;\xi',\zeta_n)|^2,\ \xi' \in R_{n-1},\ \tau = \xi_n + i\tau \in \mathbf{C}.$$

Thus $\tau^2|P^k(x;\xi',\zeta_n)|^2 \leqq CP(x,N)^2\prod_{j=1}^2[\tau^2 + (\xi_n - r_j(x,\xi'))^2]$. Choosing $\tau^2 = \Sigma_{j=1}^2(\xi_n - r_j(x,\xi'))^2$, this implies that

$$|P^k(x;\xi',\zeta_n)|^2 \leqq 4CP(x,N)^2\sum_{j=1}^2(\xi_n - r_j(x,\xi'))^2$$

The last term is $\leqq 4CP^n(x;\xi',\zeta_n,\xi',\bar\zeta_n)$ so the proof is complete.

LEMMA 2.2. *If $P(x,D)$ is of order 2, then* (1.11) *is valid.*

Proof. One may verify that the following R^k satisfy (1.1).

(2.2) $$R^k(x,\zeta,\bar\zeta) = \operatorname{Re} P^n(x,\zeta)P(x,\bar\zeta) - P^k(x,N)P^k(x,\zeta,\bar\zeta),\quad \zeta \in \mathbf{C}_n.$$

Therefore

$$\int_I \left| R^k(x';D,\bar{D})u\bar{u} \right| d\mu(\tau) \leqq \int_I \left| P^n(x';D)u \right|^2 d\mu(\tau) + \int_I \left| P^k(x';D)u \right|^2 d\mu(\tau)$$
$$+ \left| P(x',N) \right| \int_I \left| P(x';D,\bar{D})u\bar{u} \right| d\mu(\tau).$$

The first two terms are seen to satisfy (1.11) by (1.6) and Lemma 2.1. Using (2.2) with $k = n$, the last term is $\leqq \int_I | P^n(x';D)u |^2 d\mu(\tau) + \int_I | R^n(x';D,\bar{D})u\bar{u} | d\mu(\tau)$. If $R^n(x';D,\bar{D})u\bar{u} \geqq 0$, the proof is complete. But the coefficients of $R^n_{x'}$ are real symmetric, so $R^n(x';\zeta,\bar{\zeta}) = R^n(x';\xi,\xi) + R^n(x';\eta,\eta)$, $\zeta = \xi + i\eta$, and the latter terms are $\geqq 0$. Choosing $\zeta_j = D_j u(x)$, we therefore obtain $R^n(x';D,\bar{D})u\bar{u} \geqq 0$.

If $P_m(x,D)$ is strictly hyperbolic, then it is known that

$$\text{(2.3)} \qquad \sum_{|\alpha| = m-1} \int_I \left| D^\alpha u \right|^2 d\mu(\tau) \leqq \int_I R^n(x';D,\bar{D})u\bar{u}\, d\mu(\tau), \quad u \in C_0(R_n),$$

which implies (1.11). For in this case, the roots $r_j(x,\xi')$ are distinct for $\xi' \neq 0$. Thus by (1.5), $R^n(x';\xi',\zeta_n,\xi',\bar{\zeta}_n) \neq 0$, $(\xi',\zeta_n) \neq 0$. Since R^n is homogeneous of order $2(m-1)$ in (ξ',ζ_n), it follows that for any monomial $Q(\xi)$ of degree $m-1$, there is a $C > 0$ such that $|Q(\xi',\zeta_n)|^2 \leqq CR^n(x';\xi',\zeta_n,\xi',\bar{\zeta}_n)$. This implies (2.3) by arguments similar to those of §1.

The following lemma now completes the proof of Theorem 1.

LEMMA 2.3. *Condition* (B) *implies condition* (1.11).

Proof. Let $R^{k,i}$ denote the form R^k corresponding to the operator $P_{(i)}(x,D)$, $i = 1,\cdots,l$. Then he polynomials

$$R^k(x,\zeta,\bar{\zeta}) = \sum_{i=1}^{l} R^{k,i}(x,\zeta,\bar{\zeta}) \prod_{j \neq i} \left| P_{(j)}(x,\zeta) \right|^2$$

are easily seen to satisfy (1.2) for the operator $P(x,D)$. For a given $x' \in \Omega$, let $u_i = \prod_{j \neq i} P(x';D)u$. Thus $R^k(x';D,\bar{D})u\bar{u} = \sum_{i=1}^{l} R^{k,i}(x';D,\bar{D})u_i\bar{u}_i$. Then $\int_I | R^k(x';D,\bar{D})u\bar{u} | d\mu(\tau) = \int_I | \sum_{i=1}^{l} R^{k,i}(x';D,\bar{D})u_i u_i | d\mu(\tau) \leqq \sum_{i=1}^{l} \int_I R^{n,j}(x';D\bar{D})u_i u_i d\mu(\tau) = \int_I R^n(x';D,\bar{D})u\bar{u}\, d\mu(\tau)$.

It remains to prove Corollary 1. By means of the same arguments as those above, we can use (1.6) and Lemmas 1.2, 1.3 to show that the inequality (D) implies the estimate

$$\text{(2.4)} \qquad \sum_{k=1}^{m-1} \tau^{2(m-k)} \int_I \left| L_k(x,D)u \right|^2 d(\tau) \leqq C\tau^2 R(u), \quad u \in C_0^\infty(\Omega_\delta),$$

$\tau > 0$, where $C > 0$ depends on $\Omega_\delta \Subset \Omega$.

By the estimate (C), $C_1\tau^2 R(u) \leqq C_2 \int_I |P_m(x,D)u|^2 d\mu(\tau)$ which is $\leqq C_2 \int_I |P(x,D)u|^2 d\mu(\tau) + C_2 \int_I |\sum_{k=1}^{m-1} L(x, D)u|^2 d\mu(\tau)$. For τ sufficiently large, we can use (2.4) to transpose the last term, thereby obtaining the estimate (C) with $P(x,D)u$ in place of $P_m(x,D)u$.

3. We conclude by giving an example of second order operators which satisfy condition (A).

Consider a second degree real polynomial $P_2(x,\xi) = \xi_n^2 - 2\xi_n \sum_{k=1}^{n-1} b_k(x)\xi_k + \sum_{i,j=1}^{n-1} a_{ij}(x)\xi_i\xi_j$. Let $B(x) = \sum_{k=1}^{n-1} b_k(x)\xi_k$, and $A(x) = \sum_{i,j}^{n-1} a_{ij}(x)\xi_i\xi_j$. $P_2(x,\xi)$ is hyperbolic with respect to $N = (0,\cdots,0,1)$ for all $x \in \Omega$ iff the discriminant $B(x)^2 - A(x) \geqq 0$, $x \in \Omega$. In this case $\sum_{k=1}^{2} \prod_{j \neq k} |\zeta_n - r_j(x,\xi')|^2 = (\xi_n - r_1)^2 + (\xi_n - r_2)^2 + 2\tau^2 = 2(\xi_n - B(x))^2 + 2(B(x)^2 - A(x)) + 2\tau^2$ since $r_i = B(x) \pm (B(x)^2 - A(x))^{1/2}$.

Let us assume that

a) $B(x)^2 - A(x)$ depends only on ξ_i, $i = 1,\cdots, l < n-1$ and is positive definite in these variables. Thus $P_2(x,D)$ has multiple characteristics.

b) $b_j(x)$ are constant for $l < j \leqq n-1$.

Then (A) is valid. For example, $(\xi_n - B(x))^2 \leqq 2(\xi_n - B(0))^2 + 2(B(0) - B(x))^2$. But $(B(0) - B(x))^2 \leqq C(B(0)^2 - A(0))$ by assumptions a) and b).

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